

# Singularity in the discrete-time model of impacting mechanical systems

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## Abstract

It is known that many peculiar nonlinear vibration problems in impacting systems are caused by grazing incidences. Such bifurcation phenomena are normally investigated through the Poincaré map. The discrete-time map of a simple impact oscillator was derived by Nordmark, which showed that there should be a square-root singularity in the Jacobian matrix close to the grazing condition. In this paper we show that the square root singularity will be expressed only in the trace of the Jacobian matrix, while the determinant remains invariant across the grazing condition.

*Keywords:* Impact oscillator, bifurcation, grazing.

## 1 Introduction

Mechanical systems with impacts between elements occur frequently in engineering practice. Vibration problems in such systems essentially hinge on the dynamics of a moving body, possibly in a mass-spring-damper combination, impacting with a rigid stop. It is known that much of the dynamical phenomena in such systems stem from the conditions pertaining to the situation when one body just grazes the other. That is why much attention has been given to the grazing condition, and the bifurcation phenomena resulting from that [1, 2, 3, 4].

It is convenient to analyze bifurcation phenomena in any dynamical system by obtaining a discrete time model or map by the method of Poincaré surface of section. The structure of the obtained map determines the dynamics of the physical system. Thus, in understanding the dynamics of impacting systems, researchers have tried to obtain the structure of the map. It is obvious that the map should be piecewise smooth, since the map for non-impacting condition and that for the impacting condition should be different, and the two should be separated by the grazing condition. Most important in the respect is the question: What is the structure of the map in the neighborhood of a grazing orbit?

Nordmark [4] first addressed this question. He showed that in the non-impacting side the map is linear, while in the impacting side it has a square-root term. This implies that the derivative of the map approaches infinity as the grazing condition is approached from the impacting side. This results in an infinite stretching of the state space—which has come to be known as “square-root singularity.” Many researchers studied the behavior of the impact oscillator based on this map [5, 6, 7, 8].

In a two-dimensional oscillator (mass-spring-damper type), a sampling in synchronism with the external forcing function yields a two-dimensional map. Nordmark derived the condition on the whole Jacobian matrix, but not on the elements of the matrix. In the meantime, many other switching dynamical systems—most notably the power electronic circuits—were found to exhibit a new class of bifurcation that occurs when a fixed point crosses the border between two smooth regions in a piecewise smooth map. The development of

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the theory of such border collision bifurcation based on a normal form [9]. The normal form is expressed as

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{cases} \underbrace{\begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix}}_{\mathbf{J}_L} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \leq 0 \\ \underbrace{\begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}}_{\mathbf{J}_R} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \geq 0 \end{cases} \quad (1)$$

where  $\tau_L$  is the trace and  $\delta_L$  is the determinant of the Jacobian matrix  $\mathbf{J}_L$  of the system at a fixed point in one side and  $\tau_R$  is the trace and  $\delta_R$  is the determinant of the Jacobian matrix  $\mathbf{J}_R$  of the system evaluated at a fixed point in the other side.

The natural question in relation with the impact oscillator was: How does the trace and the determinant change as an impact oscillator is driven from a non-impacting state to an impacting state with the change of a parameter?

In this paper we probe this issue, and analytically prove that the determinant must be invariant while the trace alone should exhibit the square-root singularity.

## 2 Impacting Hybrid System Description

An impacting system (Fig. 1) is governed by a set of ordinary differential equations (ODEs) coupled with a set of reset maps as

$$\begin{aligned} \dot{\mathbf{x}} &= F(\mathbf{x}), & \text{if } \mathbf{x} \in S^+ \\ \mathbf{x} &\mapsto R(\mathbf{x}), & \text{if } \mathbf{x} \in \Sigma \end{aligned} \quad (2)$$

where,  $S^+ = \{\mathbf{x} : H(\mathbf{x}) > 0\}$  and  $\Sigma = \{\mathbf{x} : H(\mathbf{x}) = 0\}$ .  $H(\mathbf{x})$  is a smooth function, zero set of which defines the hard boundary  $\Sigma$ . The flow given by (2) is restricted only in the region  $S^+ \cup \Sigma$ .

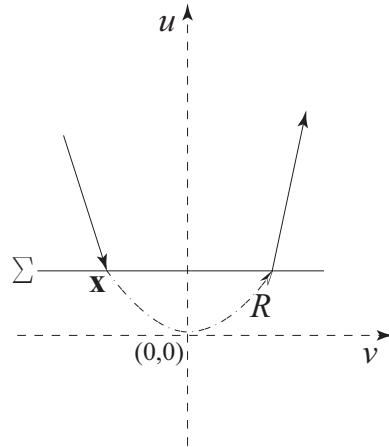


Figure 1: Upon impact, the velocity instantly reverses while the position remains the same. Thus the state instantly jumps to a new position.

Let us now define the normal velocity  $v(\mathbf{x})$  as the rate at which the trajectory approaches the impact boundary. It is given by

$$v(\mathbf{x}) := \frac{dH}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} = H_x F.$$

Similarly the normal acceleration  $a(\mathbf{x})$  of the flow with respect to the boundary is

$$a(\mathbf{x}) := (H_{\mathbf{x}} F)_{\mathbf{x}} F.$$

We may now be more specific about the form the reset map  $R(\mathbf{x})$  takes. To that end, we observe that the reset map has to be a smooth function of the normal velocity  $v(\mathbf{x})$  and furthermore  $R$  maps to itself when grazing occurs. Since at grazing the normal velocity with respect to the boundary becomes zero ( $v(\mathbf{x}) = 0$ ), the reset map can be formulated as

$$R(\mathbf{x}) = \mathbf{x} + W(\mathbf{x})v(\mathbf{x}) \quad (3)$$

where  $W$  is a smooth  $2 \times 1$  matrix.

### 3 Grazing and Discontinuity Mapping

Grazing occurs when a trajectory becomes tangent to the discontinuity boundary  $\Sigma$ , as shown in Fig. 2. A point  $\mathbf{x} = \mathbf{x}^*$  is called a regular grazing point if it satisfies the conditions

$$H(\mathbf{x}^*) = 0$$

$$v(\mathbf{x}^*) = 0$$

$$a(\mathbf{x}^*) = a^* > 0$$

In addition the scalar function  $H(\mathbf{x})$  is assumed to be well defined at  $\mathbf{x} = \mathbf{x}^*$ , i.e.,  $H_{\mathbf{x}}(\mathbf{x}^*) \neq 0$ .

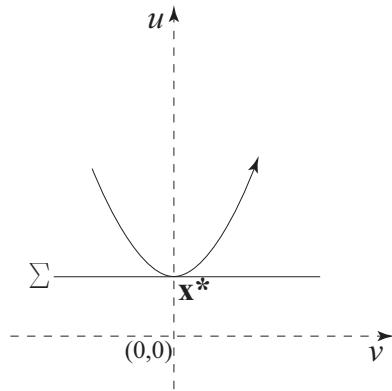


Figure 2: Grazing of discontinuity boundary.

For the part of the flow that does not have any impact with the discontinuity boundary the mapping is given by the ODE only. Whenever there is an impact with the boundary, the reset map comes into action and there is a discontinuity in the flow. The discontinuity near grazing is of particular interest. Special kinds of mapping have been proposed to account for this discontinuity [10]. In this present case the zero-time discontinuity mapping (ZDM) is dealt with.

Let us consider the situation as shown in Fig. 3. There is an orbit which grazes the discontinuity boundary  $\Sigma$  at a point  $\mathbf{x}^*$  at some point of time  $t_0$ . Now let there be another trajectory  $(\mathbf{x}_0 \mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_4)$  close to the grazing orbit. Let us back-trace the trajectory, governed by the ODE as in (2), from the point  $\mathbf{x}_2$  to the point  $\mathbf{x}_3$  such that the time taken by the trajectory to reach from  $\mathbf{x}_0$  to  $\mathbf{x}_2$  is the same as would have been taken by

the flow to reach from  $\mathbf{x}_3$  to  $\mathbf{x}_2$ . Thus, we can consider the systems' dynamics as if the switching boundary were not there. In that case we have to assume an instantaneous jump of the state from  $\mathbf{x}_0$  to  $\mathbf{x}_3$ . The ZDM is defined as the mapping  $\mathbf{x}_0 \mapsto \mathbf{x}_3$ .

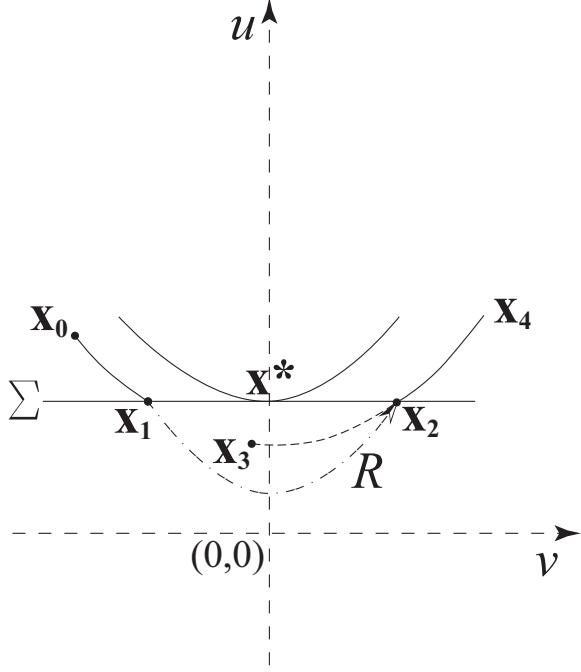


Figure 3: Zero-time discontinuity mapping near grazing.

Let  $\phi(\mathbf{x}_0, t)$  be the flow obtained as the solution of the ODE starting from  $\mathbf{x}_0$ , i.e.  $\phi(\mathbf{x}_0, 0) = \mathbf{x}_0$ . It has been shown in [10] that the form of the ZDM, excluding higher order terms, is

$$\mathbf{x}_3 = \mathbf{x}_0 - W^*(\sqrt{2a^*})y \quad (4)$$

where,  $W^* = W(\mathbf{x}^*)$ , and  $y = \sqrt{-H_{\min}(\mathbf{x}_0)}$ .  $H_{\min}(\mathbf{x}_0)$  is defined as the minimum value of  $H(\phi(\mathbf{x}_0, t))$  with the smallest  $|t|$ , i.e., the lowest point that the trajectory would have reached if the switching boundary were not there. Obviously for the situation as described in Fig. 3,  $H_{\min}(\mathbf{x}_0)$  will be negative except for the case when  $\mathbf{x}_0$  is the same as  $\mathbf{x}^*$ .

Now, let us consider a periodic orbit which has an intersection with the discontinuity boundary very close to the grazing orbit, as shown in Fig. 4. The stroboscopic Poincaré map in this case is  $P_s = P_2 \circ ZDM \circ P_1$ , where  $P_1$  is the map that takes a point on the Poincaré plane and maps it to the discontinuity boundary  $\Sigma$  by evolution through the ODE in (2), and  $P_2$  is the map that takes a point on the discontinuity boundary  $\Sigma$  and maps it back to the Poincaré plane via the ODE. The form of this stroboscopic map can be derived, in first order approximation, as:

$$\begin{aligned} P_1 & : \mathbf{x} \mapsto N_1 \mathbf{x} \\ ZDM \circ P_1 & : \mathbf{x} \mapsto N_1 \mathbf{x} - \sqrt{2a^*} \sqrt{-H_{\min}(N_1 \mathbf{x})} W^* \\ P_2 \circ ZDM \circ P_1 & : \mathbf{x} \mapsto N_2 N_1 \mathbf{x} - \sqrt{2a^*} \sqrt{-H_{\min}(N_1 \mathbf{x})} N_2 W^* \end{aligned} \quad (5)$$

where,  $N_1 := \frac{dP_1}{dx}|_{x=x_0}$ ,  $N_2 := \frac{dP_2}{dx}|_{x=x_0}$ .

On the Poincaré plane, we define  $x_0$  as the origin, and the stroboscopic map in this case is defined as the map which takes the initial deviation on the Poincaré plane,  $(\mathbf{x} - \mathbf{x}_0)$ , and maps this deviation again on the Poincaré plane. Since  $\mathbf{x}_0$  maps to the point  $N_2 N_1 \mathbf{x}_0$ , this map takes the form

$$(\mathbf{x} - \mathbf{x}_0) \mapsto N_2 N_1 (\mathbf{x} - \mathbf{x}_0) - \sqrt{2a^*} \sqrt{-H_{\min}(N_1 \mathbf{x})} N_2 W^* \quad (6)$$

Also  $H_{\min}(N_1 \mathbf{x})$  is linearized about grazing to the form

$$H_{\min}(N_1 \mathbf{x}) = H_{\mathbf{x}} N_1 (\mathbf{x} - \mathbf{x}_0) + H.O.T$$

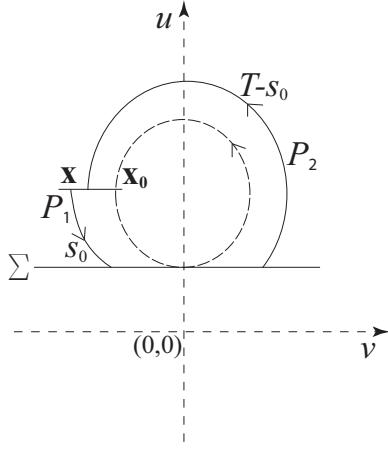


Figure 4: A grazing periodic orbit and stroboscopic map.

As can be seen from the expression of the stroboscopic map, and also of ZDM alone, a square-root term  $\sqrt{-H_{\min}}$  is present which accounts for the square-root singularity when the Jacobian of the map is considered. The next section deals with this Jacobian and the square-root singularity therein.

## 4 Investigating the Trace and the Determinant of the Jacobian for the Singularity

The Jacobian of the stroboscopic map near grazing would be

$$J = N_2 \circ J_{ZDM} \circ N_1 \quad (7)$$

where  $J_{ZDM}$  is the Jacobian of the ZDM given by

$$J_{ZDM} = \frac{\partial f}{\partial \mathbf{x}} = I + \sqrt{2a^*} \cdot \frac{W^* H_{\mathbf{x}}}{2\sqrt{-H_{\min}}}. \quad (8)$$

To arrive at the particular forms  $W^*$  and  $H_{\mathbf{x}}$  would take, let us concentrate on the one degree-of-freedom impact oscillator (shown in figure 5). The mass (assumed to be unity without any loss of generality) is tied with a spring-damper element and is acted upon by an external force  $g(t)$ . At a distance  $\sigma$  from the mass there is an impacting wall so that for  $u < \sigma$  the motion of the mass is governed by second-order differential equation

$$\frac{d^2 u}{dt^2} + 2\zeta\omega_n \frac{du}{dt} + \omega_n^2 u = g(t), \text{ for } u < \sigma \quad (9)$$

and at  $u = \sigma$  the reset map  $R$  is applied. This system is a two-dimensional system, i.e.  $\mathbf{x} \in \mathbb{R}^2$ . Considering the velocity of motion of the mass as  $v = \frac{du}{dt}$ , the state vector can be written as  $\mathbf{x} = (u, v)^T$ .

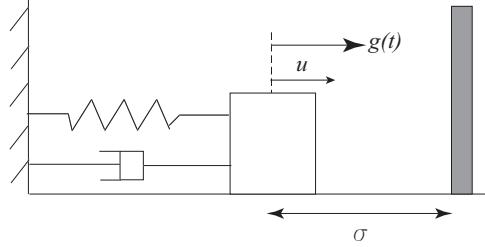


Figure 5: A one degree-of-freedom impact oscillator.

The equation of the discontinuity boundary  $\Sigma$  in the present case is

$$H(\mathbf{x}) = H(u, v) = \sigma - u.$$

Thus we have  $\frac{\partial H}{\partial v} = 0$ , and hence,

$$H_{\mathbf{x}} = (h_1 \ 0). \quad (10)$$

where  $h_1 = \frac{\partial H}{\partial u}$ .

The reset map is  $R : \Sigma \mapsto \Sigma$ , where  $R(\mathbf{x})$  has the form given in (3). Note that the position  $u$  does not vary during the impact, i.e., the position of the mass just before the impact,  $u^-$ , is same as that just after the impact,  $u^+$ , while the velocity of motion  $v$  changes. Thus from (3)

$$W = (0 \ w_2)^T; \quad (11)$$

where  $w_2$  is the constant of restitution.

Using (10) and (11) in (8) we obtain

$$\begin{aligned} J_{ZDM} &= I + \sqrt{2a^*} \cdot \frac{W^* H_{\mathbf{x}}}{2\sqrt{-H_{\min}}} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sqrt{2a^*}}{2\sqrt{-H_{\min}}} \cdot \begin{pmatrix} 0 \\ w_2 \end{pmatrix} \begin{pmatrix} h_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sqrt{2a^*}}{2\sqrt{-H_{\min}}} \begin{pmatrix} 0 & 0 \\ w_2 h_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \end{aligned} \quad (12)$$

where

$$\alpha = \frac{w_2 h_1 \sqrt{2a^*}}{2\sqrt{-H_{\min}}}$$

#### 4.1 Investigating the Determinant for Singularity

From (7), the determinant of the normal form map near grazing is

$$\begin{aligned} |J| &= |N_2| |J_{ZDM}| |N_1| \\ &= |N_2| |N_1| \end{aligned}$$

since  $|J_{ZDM}| = 1$ , from (12).

Since the singularity is only in the ZDM, and not in the maps  $N_1$  and  $N_2$ , we conclude that the determinant of the normal form map does not contain the square-root singularity, and remains invariant in the immediate neighborhood of the grazing orbit.

## 4.2 Investigating the Trace for Singularity

To obtain the expression for the trace of the Jacobian  $J$  in (7), we need to obtain first the expressions for the maps  $P_1$  and  $P_2$ .

Let us consider a periodic solution to the equation given in (9) as

$$p(t) = (u(t), v(t))^T.$$

Let  $(p(t) + \delta p(t))$  be a perturbed orbit, where  $\delta u$  satisfies the following variational equation

$$\delta \ddot{u} + 2\zeta \omega_n \delta \dot{u} + \omega_n^2 \delta u = 0. \quad (13)$$

The variational equation needs to be solved to obtain the perturbed flow  $\delta p(\tau) = (\delta u(\tau), \delta v(\tau))^T$ . Solving the variational equation amounts to solving the first-order differential equations

$$\frac{d}{dt} \begin{pmatrix} \delta u(t) \\ \delta v(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta \omega_n \end{pmatrix} \begin{pmatrix} \delta u(t) \\ \delta v(t) \end{pmatrix},$$

with  $\delta u(0) = \delta u_0, \delta v(0) = \delta v_0$ .

The solution of the above problem can be expressed as

$$\begin{pmatrix} \delta u(\tau) \\ \delta v(\tau) \end{pmatrix} = N_\tau \begin{pmatrix} \delta u_0 \\ \delta v_0 \end{pmatrix}$$

where

$$N_\tau = e^{-\zeta \omega_n \tau} \begin{pmatrix} \cos(\omega_0 \tau) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_0 \tau) & \sin(\omega_0 \tau)/\omega_0 \\ -\frac{1}{\sqrt{1-\zeta^2}} \omega_0 \sin(\omega_0 \tau) & \cos(\omega_0 \tau) - \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_0 \tau) \end{pmatrix} \quad (14)$$

with  $\omega_0 = \omega_n \sqrt{1 - \zeta^2}$ .

Now we can proceed to obtain the expression of the trace of the Jacobian in (7). In the situation shown in Fig. 4, the flow takes time  $s_0$  to reach the discontinuity boundary starting from the Poincaré plane, and time  $(T - s_0)$  to return to the Poincaré plane starting from the discontinuity boundary, where  $T$  is the time period of the external forcing function  $g(t)$ . Thus using the notation in (14),

$$N_1 = N_{s_0} = e^{-\zeta \omega_n s_0} \begin{pmatrix} n_{11} & n_{12} \\ n_{13} & n_{14} \end{pmatrix} \quad (15)$$

$$N_2 = N_{(T-s_0)} = e^{-\zeta \omega_n (T-s_0)} \begin{pmatrix} n_{21} & n_{22} \\ n_{23} & n_{24} \end{pmatrix} \quad (16)$$

where the expressions for  $n_{11}, n_{12}, n_{13}, n_{14}, n_{21}, n_{22}, n_{23}$  and  $n_{24}$  can be derived from (14).

Using (12), (15) and (16) in (7), we get

$$\begin{aligned} J &= e^{-\zeta \omega_n T} \begin{pmatrix} n_{21} & n_{22} \\ n_{23} & n_{24} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} \\ n_{13} & n_{14} \end{pmatrix} \\ &= e^{-\zeta \omega_n T} \begin{pmatrix} n_{21} + \alpha n_{22} & n_{22} \\ n_{23} + \alpha n_{24} & n_{24} \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} \\ n_{13} & n_{14} \end{pmatrix} \\ &= e^{-\zeta \omega_n T} \begin{pmatrix} n_{21}n_{11} + n_{22}n_{13} + \alpha n_{22}n_{11} & * \\ * & n_{23}n_{12} + n_{24}n_{14} + \alpha n_{24}n_{12} \end{pmatrix} \\ \Rightarrow \text{Tr}(J) &= e^{-\zeta \omega_n T} \{ n_{21}n_{11} + n_{22}n_{13} + n_{23}n_{12} + n_{24}n_{14} + \alpha \boxed{(n_{22}n_{11} + n_{24}n_{12})} \} \end{aligned} \quad (17)$$

The expression for the trace of the Jacobian  $\text{Tr}(J)$ , in (17), shows that the singularity term  $\alpha$  has a coefficient  $e^{-\zeta\omega_n T}(n_{22}n_{11} + n_{24}n_{12})$ . Let us take a closer look at this coefficient. Using (14), (15) and (16)

$$\begin{aligned} n_{11} &= \cos(\omega_0 s_0) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_0 s_0) \\ n_{12} &= \frac{\sin(\omega_0 s_0)}{\omega_0} \\ n_{22} &= \frac{\sin\{\omega_0(T-s_0)\}}{\omega_0} \\ \text{and } n_{24} &= \cos\{\omega_0(T-s_0)\} - \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\{\omega_0(T-s_0)\}. \end{aligned}$$

Therefore it follows that

$$\boxed{n_{22}n_{11} + n_{24}n_{12} = \frac{\sin(\omega_0 T)}{\omega_0} \neq 0}, \quad \forall \omega_0 \neq \frac{m\omega_{\text{forcing}}}{2} \quad (18)$$

where  $\omega_{\text{forcing}}$  is the angular frequency of the periodic forcing function  $g(t)$ , i.e.,  $\omega_{\text{forcing}}T = 2\pi$ , and  $m \in I$ . Thus the coefficient of  $\alpha$  in the expression of the trace (17) of the Jacobian of the stroboscopic map must be a non-zero entity. Thus the singularity in  $\alpha$  survives, and hence a square-root singularity must occur in the trace of the Jacobian.

## 5 Conclusions

In this paper we have probed the variation of the trace and determinant of the Jacobian matrix of map of a hard-impact oscillator as it goes from non-impacting state to an impacting state. We have shown that the square-root singularity should be expressed only in the trace of the Jacobian matrix while the determinant should remain invariant in the immediate neighborhood of a grazing orbit.

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